Real Analysis Study Note

Minjun Kim
mikanebi0@yonsei.ac.kr   mikane.net

July 6, 2025

Preliminaries: Topological Spaces

Definition 1. **A topology** on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following properties.

1. $\{\emptyset, \mathcal{T}\} \subset \mathcal{T}$
2. If $U_i \in \mathcal{T}$ for $1 \leq i \leq n$ then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.
3. If $U_\alpha \in \mathcal{T}$ for $\alpha \in I$ where I is an arbitrary index set then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

Definition 2. If a topology $\mathcal{T}$ is defined on a set X, we say $(X, \mathcal{T})$ **a topological space.**

Definition 3. Let $\mathcal{T}$ and $\mathcal{T}'$ be a topology on a set X.

- $\mathcal{T}'$ is finer than $\mathcal{T}$ if $\mathcal{T}' \subset \mathcal{T}$.
- $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ if $\mathcal{T}' \subset \mathcal{T}$ and $\mathcal{T} \neq \mathcal{T}'$.

Definition 4. **A basis** on a topological space $(X, \mathcal{T})$ is a collection $\mathcal{B}$ of subsets of X satisfying the following properties.

1. For every $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
2. Let $B_1, B_2 \in \mathcal{B}$. If $x \in B_1 \cap B_2$ then $B_1 \cap B_2 \in \mathcal{B}$.

Definition 5. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

- A set A is **open** if $A \in \mathcal{T}$.
- A set A is **closed** if $X \setminus A \in \mathcal{T}$.
- **The interior** of A, denoted by A° is the union of all open subset of A.
- **The closure** of A, denoted by $\overline{A}$ is the intersection of all closed set containing A.
- **The boundary** of A, denoted by ∂A is $\overline{A} \cap \overline{X \setminus A}$.

- A is **dense** if $\overline{A} = X$.
- A is **nowhere dense** if $\left(\overline{A}\right)^\circ = \emptyset$
- x is a **limit point** of A if for every open set U containing x, $A \cap (U \setminus \{x\}) \neq \emptyset$.

Let X, Y and Z be topological spaces.

Definition 6. A function $f : X \to Y$ is continuous if

$$\text{for every open subset } U \subset Y,\ f^{-1}(U) \text{ is open.}$$

Theorem 7. Assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. If $h = g \circ f$ then $h : X \to Z$ is continuous.

Measurable Spaces

Definition 8. A σ-algebra of a set X is a collection $\mathfrak{M}$ of subsets of a set X satisfying the following properties.

1. $\emptyset, X \in \mathfrak{M}$.
2. If $A \in \mathfrak{M}$, then $X \setminus A \in \mathfrak{M}$.
3. If $A_i \in \mathfrak{M}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.

Definition 9. For a set X, if the σ-algebra $\mathfrak{M}$ is defined on X, we call $(X, \mathfrak{M})$ a **measurable space**. We also say that the elements of $\mathfrak{M}$ are $\mathfrak{M}-$**measurable sets**.

Let $(X, \mathfrak{M})$ be a mesurable space.

Theorem 10. If $A_i \in \mathfrak{M}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathfrak{M}$.

Proof. Obvious. □

Theorem 11. If $A, B \in \mathfrak{M}$ then $A \setminus B \in \mathfrak{M}$.

Proof. Since $A, \mathfrak{M} \setminus B \in \mathfrak{M}$, $A \setminus B = A \cap (\mathfrak{M} \setminus B) \in \mathfrak{M}$. □

Definition 12. Let X and Y be topological spaces. A mapping $f : X \to Y$ is **measurable** if

$$\text{for every open set } V \in Y,\ f^{-1}(V) \text{ is a mesurable set in } X.$$

Let X be a measurable space and Y, Z be topological spaces.

Theorem 13. Assume that $f : X \to Y$ is measurable and $g : Y \to Z$ is continuous. Let $h = g \circ f$ then $h : X \to Z$ is measurable.

Proof. Let $U \subset Z$ be an open set.

$$h^{-1}(U) = f^{-1}(g^{-1}(U))$$

is a measurable set since $g^{-1}(U)$ is open and f is measurable. $\square$