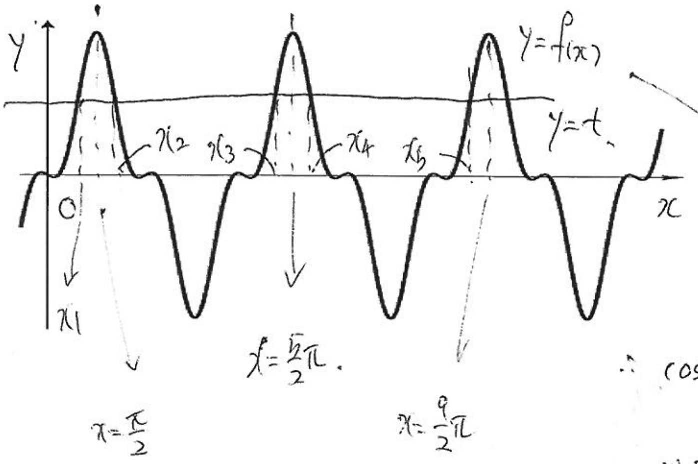
* 2020학년도 평가원 6월 수학 가형 30번.

$f(x) = a\sin^3 x + b\sin x$

$f(\frac{\pi}{4}) = 3\sqrt{2}$, $f(\frac{\pi}{3}) = 5\sqrt{3}$

$\rightarrow a \times \frac{2\sqrt{2}}{8} + b \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}a + \frac{\sqrt{2}}{2}b = 3\sqrt{2} \rightarrow a + 2b = 12$

$a \times \frac{3\sqrt{3}}{8} + b \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}a + \frac{\sqrt{3}}{2}b = 5\sqrt{3} \rightarrow 3a + 4b = 40$.



$\therefore a = 16,\ b = -2.\ f(x) = 16\sin^3 x - 2\sin x$

$\rightarrow$ $\sin x$로만 구성된 연산함수 $\rightarrow$ 주기 존재(2π)

$f'(x) = 48\sin^2 x\cos x - 2\cos x = 2\cos x(24\sin^2 x - 1)$

$\therefore \cos x = 0$ or $\sin^2 x = \frac{1}{24}$일 때, $f'(x)$의 부호가

바뀌고, $f(x)$는 극값을 갖는다.

(i) $\cos k_1 = 0$이면 $\sin k_1 = \pm 1$. $f(k_1) = 16 \times (\pm 1)^3 - 2 \times (\pm 1) = \pm 14$

$\left(\left| \mp \frac{4}{3\sqrt{24}} \right| < 1 \right)$

(ii) $\sin k_2 = \pm \frac{1}{\sqrt{24}}$이면 $f(k_2) = 16 \times \left(\pm \frac{1}{\sqrt{24}}\right)^3 - 2 \times \left(\pm \frac{1}{\sqrt{24}}\right) = \pm \frac{2}{3\sqrt{24}} - \left(\pm \frac{6}{3\sqrt{24}}\right) = \mp \frac{4}{3\sqrt{24}}$.

수열 x_n의 특징을 정리해보면

$x_1 + x_2 = \pi$, $x_3 + x_4 = 5\pi$, $x_5 + x_6 = 9\pi$, $\cdots$ / $x_1 + 2\pi = x_3$, $x_2 + 2\pi = x_4$, $x_3 + 2\pi = x_5$, $\cdots$

따라서 n이 자연수일 때, $\begin{cases} x_{2n-1} + x_{2n} = 4n\pi - 3\pi. \\ x_{2n+1} = x_{2n-1} + 2\pi, \quad x_{2n+2} = x_{2n} + 2\pi \end{cases}$

이 때, $c_n = \int_{3\sqrt{2}}^{5\sqrt{3}} \frac{t}{f'(x_n)} dt = \frac{1}{f'(x_n)} \times \int_{3\sqrt{2}}^{5\sqrt{3}} t\,dt = \frac{1}{f'(x_n)} \times \left[\frac{t^2}{2}\right]_{3\sqrt{2}}^{5\sqrt{3}} = \frac{1}{f'(x_n)} \times \frac{57}{2}$.

$\therefore \begin{cases} c_1 = c_3 = c_5 = \cdots, \ c_2 = c_4 = c_6 = \cdots \ (\text{주기성}) \\ c_1 + c_2 = c_3 + c_4 = c_5 + c_6 = \cdots = 0 \ (\text{대칭성}) \end{cases}$

따라서 $\sum_{n=1}^{101} c_n = 50 \times (c_1 + c_2) + c_1 = c_1 = \int_{3\sqrt{2}}^{5\sqrt{3}} \frac{t}{f'(x_n)} dt = p + q\sqrt{2}$.

$f(x_1) = 16\sin^3 x_1 - 2\sin x_1 = t.$

$\therefore f'(x_1)\,dx_1 = dt$ 에서 $f'(x_1) = \dfrac{dt}{dx_1}$ ($\because$ x_1과 t 값은 서로 간에 종속적이다)

$$\therefore \int_{3\sqrt{2}}^{5\sqrt{3}} \frac{t}{f'(x_1)}\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} t\,dx_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x_1)\,dx_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x)\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \{16\sin^3 x - 2\sin x\}\,dx.$$ (문제 조건에 나온 $f(\frac{\pi}{4}) = 3\sqrt{2}$, $f(\frac{\pi}{3}) = 5\sqrt{3}$ 을 통해 밑끝, 윗끝 확인 가능)

* $\int \sin^3 x\,dx$ 는 $\cos x = s$ 로 치환하면 $-\sin x\,dx = ds$ 이므로

$$\int \sin x \cdot \sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx = -\int \{1-s^2\}\,ds$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 16\sin^3 x\,dx = -16\times\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \{1-s^2\}\,ds = -16\times\left[s - \frac{s^3}{3}\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}$$

$$= -16\times\left\{\left(\frac{1}{2}-\frac{1}{24}\right)-\left(\frac{\sqrt{2}}{2}-\frac{2\sqrt{2}}{24}\right)\right\} = -16\times\left(\frac{11}{24}-\frac{10\sqrt{2}}{24}\right) = -\frac{22}{3}+\frac{20\sqrt{2}}{3}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -2\sin x\,dx = 2\times\left[\cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = 2\times\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right) = 1-\sqrt{2}.$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \{16\sin^3 x - 2\sin x\}\,dx = -\frac{22}{3}+\frac{20\sqrt{2}}{3}+1-\sqrt{2} = -\frac{19}{3}+\frac{17\sqrt{2}}{3} \quad //$$

($\therefore q - p = 12$)