Appendix B

Thomas Precession

B.1 Thomas Precession in Cartesian Coordinates

Let three different system like fallowing figure:

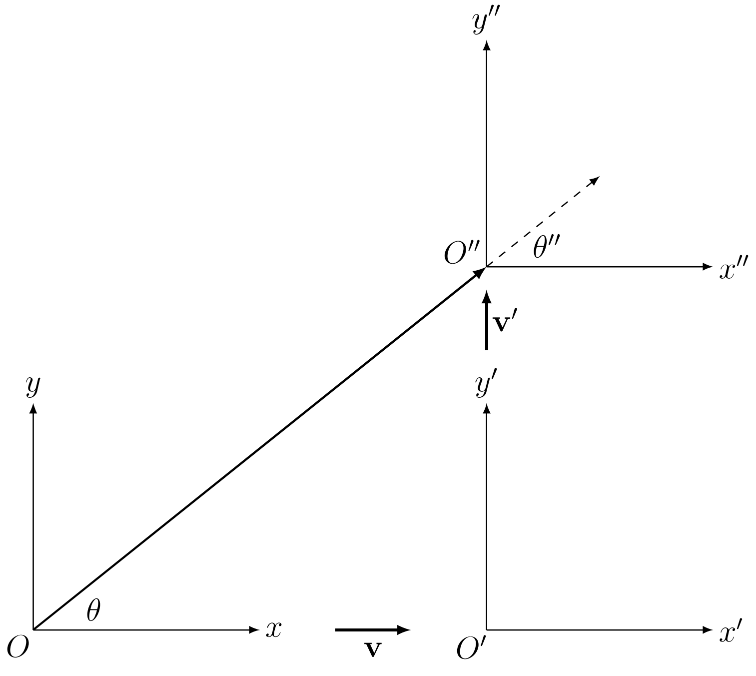

Figure B.1: Lorentz transformation in three systems $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$

Consider these three systems. Suppose that system $\mathcal{S}$(x-y coordinate) is stopped, and system $\mathcal{S}'$(x'-y' coordinate) and $\mathcal{S}''$(x''-y'' coordinate) each moves to right and above side. The line from origin O of $\mathcal{S}$ to origin O'' of $\mathcal{S}''$ making an angle θ in $\mathcal{S}$ and an angle $\mathcal{S}''$.

The angle θ and θ' In non-relativistic mechanics, they must be same.(Because the motion of coordinates only has translation, not a rotation) Then, how about considering relativistic effects? We can't guarantee about it.[1] So, you want to check what happen in relativistic-mechanics, applying Lorentz transformation to each coordinates.

Lorentz Transformation We can calculate the angles in two frames by applying the Lorentz transformation in each frame.

In x' coordinates,

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx/c^2) \\ y' &= y \end{aligned} \tag{B.1}$$

In x coordinates,

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma(t' + vx'/c^2) \\ y &= y' \end{aligned} \tag{B.2}$$

In x'' coordinates,

$$\begin{aligned} y'' &= \gamma'(y' - v't') \\ t'' &= \gamma'\left(t' - vy'/c^2\right) \\ x'' &= x' \end{aligned} \tag{B.3}$$

And also, it satisfies fallowing relation, too.[2]

$$\begin{aligned} y' &= \gamma'(y'' + v't'') \\ t' &= \gamma'\left(t'' + vy''/c^2\right) \\ x' &= x'' \end{aligned} \tag{B.4}$$

Now, combine all of these relations, we get fallowing relation.

$$\begin{aligned} y'' &= \gamma'\left[y - v'\gamma(x - vt)\right] \\ x'' &= \gamma(x - vt) \end{aligned} \tag{B.5}$$

Then, form these results, we get angle θ like:

$$\tan\theta = \frac{y}{x} = \frac{y'}{vt} = \left.\frac{\gamma'\left(y'' + v't'\right)}{vt}\right|_{y''=0} = \frac{\gamma' v' t'}{vt} \tag{B.6}$$

$$t = \gamma\left(t' + vx'/c^2\right)\big|_{x''=x'=0} = \gamma\gamma'\left(t'' + v'y''/c^2\right)\big|_{y''=0} = \gamma\gamma' t'' \tag{B.7}$$

So that

$$\tan\theta = \frac{\gamma' v' t''}{v\gamma\gamma' t''} = \frac{v'}{\gamma v} \tag{B.8}$$

[1] Lorentz contraction wil occur multiply.

[2] Of course, γ and γ' each satisfy $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\gamma' = 1/\sqrt{1 - v'^2/c^2}$

Then, with the same way we get another angle θ'', too.

$$\tan\theta'' = \frac{y''}{x''} = \frac{\gamma'\left[y' - v't'\right]}{x'} \tag{B.9}$$

Remember that x'' and y'' are the coordinate of origin O in system $\mathcal{S}$ in the $\mathcal{S}''$ system, so it satisfies fallowing relation.

$$\tan\theta'' = \left.\frac{\gamma'\left[y' - v't'\right]}{x'}\right|_{y=0} = -\frac{\gamma' v't'}{x'} = -\left.\frac{\gamma' v't'}{\gamma(x - vt)}\right|_{x=0} = \frac{\gamma' v't'}{\gamma vt} \tag{B.10}$$

$$t' = \gamma(t - vx/c^2)\big|_{x=0} = \gamma t \tag{B.11}$$

So that

$$\tan\theta'' = \frac{\gamma' v'}{v} \tag{B.12}$$

Now we get all of the conclusion effected by Lorentz transformation. As we suspected, these two angle are not same. Then, what it mean? From the result of relativistic effect, the system $\mathcal{S}''$ are little rotated. In other words, the relativistic effect makes Precession.[3] Then, remind that the net net rotation only effected by relativistic mechanics are not θ''. That, Thomas Angle[4] is given by fallowing relation.

$$\theta_T = \theta'' - \theta = \tan^{-1}\left(\frac{\gamma' v'}{v}\right) - \tan^{-1}\left(\frac{v'}{\gamma v}\right) \tag{B.13}$$

Choose $v'/v \ll 1$, formula (B.13) is approximately

$$\theta_T \approx \left(\gamma' - \frac{1}{\gamma}\right)\frac{v'}{v} \approx \left(\gamma' - \frac{1}{\gamma}\right)\theta \tag{B.14}$$

In here, remind that $\tan\theta = v'/v$ and the angle θ is very small.

[3]We call this 'Thomas Precession'.

[4]For convenience, I define that 'the net rotation angle by relativistic effect' a 'Thomas Angle'.